Global existence and gradient estimates for some quasilinear parabolic equations (Nonlinear Evolution Equations and Applications)

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Global existence and gradient estimates for some quasilinear parabolic equations

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1 Introduction

In this talk we treat the initial-boundary value problem for some types of quasilinear parabolic equations of the form:

$$u_t - div\{\sigma(|\nabla u|^2)\nabla u\} + g(u, \nabla u) = 0 \qquad x \in \Omega,\ t > 0 \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega; \quad u(x, t) = 0, \quad x \in \partial\Omega,\ t > 0 \tag{1.2}$$

where Ω is a bounded domain in R^N with a smooth, say, C^2 boundary $\partial\Omega$.

We are interested in smoothing effects near $t = 0$ and asymptotic behaviours as $t \to \infty$ as well as the global existence of solutions. We would emphasize that our perturbations $g(u, \nabla u)$ heavily depends on ∇u, which is different from the usual ones $g(u)$.

First, we consider the case $\sigma(|\nabla u|^2) = |\nabla u|^m, m \geq 0$, and $g(u, \nabla u) = \mathbf{b}(u) \cdot \nabla u$ with $|b(u)| \leq k_1|u|^\beta$.

In this case the principal term is often called m-Laplacian and the perturbation describes a convection effect with velocity field $\mathbf{b}(u)$. Concerning the initial data we only assume $u_0 \in L^q(\Omega), q \geq 1$, while we want to derive estimates for $\|\nabla u(t)\|_\infty, t > 0$. So, nonlinear semigroup theory , if it could be applied to our equations, would not be sufficient for our purpose. We carry out careful analysis based on Gagliardo-Nirenberg inequality to derive desired apriori estimates . Our results seems to be new even for the unperturbed ,most standard equation with $\mathbf{b}(u) = 0$ (cf. Alikakos and Rostamian [1], Nakao [11]).

Secondly, we consider the case : $\sigma = |\nabla u|$ and $g = \pm|\nabla u|^{1+\beta}, \beta > m$. This perturbation is stronger than usual ones $g = g(u)$ and to prove the global existence of solutions careful gradient estimates are essentially required. Further, to gurantee the convergence of approximate solutions we need some estimates for second order derivatives , which is an essential difference from the case $g = g(u)$. We can compare our results with some known results for the case $g = g(u)$ which not necesarily monotone increasing (cf. M.Tsutsumi [20], M.Otani [18],H.Ishii [8], M.Nakao [12,13, Y.Ohara [16,17] etc.) The third problem we consider is the case:$\sigma = 1/\sqrt{1+|\nabla u|^2}$ and $g = \pm|\nabla u|^{1+\beta}, \beta > 0$. In this case the principal term is no longer coercive and hence, to controle the perturbation is more delicate. Of course, if we assume that the initial data is sufficiently smooth and small, it is not difficult to prove global existence of smooth and small amplitude solutions. But, we want to treat not so smooth initial data. In fact, we prove that if u_0 belongs to W_0^{1,p_0} with a certain $p_0 > 0$ and $\|\nabla u_0\|_{1,p_0}$ is sufficiently small , then there exists a unique global solution in some class, satisfying

$$\|\nabla u(t)\|_\infty \leq Ct^{-\xi}e^{-\lambda t}$$

with $\xi = N/(2p_0 - 3N)$ and some $\lambda > 0$.

Our result is a generalization of our recent work [15] where nonperturbed equation is considered. There are many intersting papers treating quasilinear parabolic equations of the mean curvature type (N.Trudinger [19], C.Gerhardt [7], K.Ecker [5], G.Lieberman [10] etc.). But, no result concerning smoothing effect seems to be known for the equation with a strong perturbation $|\nabla u|^p$, power nonlinearity of ∇u.

Almost throughout the paper we assume that the mean curvature $H(x)$ of the boundary $\partial\Omega$ is nonpositive with respect to the outward normal. This is essentially used to derive a priori estimastes for $\|\nabla u(t)\|_p, p >> 1$.

This talk is based on my joint works [3,4,14] with Caisheng Chen (Hohai Univ., Nanjing, P.R. China) and Y.Ohara (Yatsusiro College of Technology, Yatsusiro, Kumamoto).

2 Statement of results

We first consider :

Case 1. $\sigma(|\nabla u)|^2) = |\nabla u|^m, m \geq 0$, and $g(u, \nabla u) = \mathbf{b}(u) \cdot \nabla u$ with

$$|\mathbf{b}(u)| \leq k_0|u_0|^\beta, \beta > m.$$

We begin with existence and estimates for $\|\nabla u(t)\|_{m+2}$ and $\|u(t)\|_\infty$.

Theorem 1 *[14] Let $u_0 \in L^q, q \geq 1$. Then, the problem (1.1)-(1.2) admits a unique solution $u(t)$ in the class*

$$L^\infty_{loc}((0,\infty); W_0^{1,m+2}) \bigcap W^{1,2}_{loc}((0,\infty); L^2) \bigcap C(R^+; L^1) \bigcap L^\infty(R^+; L^q),$$

satisfying

$$\|\nabla u(t)\|_{m+2} \leq C_0(1+t)^{-1/m}(1+t^{-\mu}), \quad t > 0 \tag{2.1}$$

and

$$\|u(t)\|_\infty \leq C_0(1+t)^{-1/m}(1+t^{-\lambda}), \quad t > 0 \tag{2.2}$$

where we set

$$\lambda = N/(mN + q(m+2)),$$

$$\alpha = (2\beta - m - mq/N)$$

and

$$\mu = \frac{1 + 2(\alpha - 1)^+ + (2-q)^+\lambda}{m+2}.$$

To derive estimates for $\|\nabla u(t)\|_\infty$ we need the following important assumption

Hyp.A. When $N \geq 2, \partial\Omega$ is of C^2-class and the mean curvature $H(x)$ of $\partial\Omega$ at $x \in \partial\Omega$ with respect the outward normal is nonpositive.

Theorem 2 *[14] Under the hypothesis Hyp.A the solutions $u(t)$ in Theorem 1 belong further to $L^\infty_{loc}((0,\infty); W_0^{1,\infty}$ and satisfy the estimates*

$$\|\nabla u(t)\|_\infty \leq C(1+t)^{-\tilde{\nu}}(1+t^{-\xi}), \quad t > 0$$

if $\alpha \leq 1$, and

$\|\nabla u(t)\|_\infty \leq C_\varepsilon(1+t)^{-\tilde{\nu}}(1+t^{-\xi-\varepsilon}) \quad t > 0)$ *if $\alpha \geq 1$ and $\mu < \alpha - 1 + (m+2)(N\alpha+2)/mN$,*

where

$$\xi = \frac{2\mu + N max\{1,\alpha\}}{mN + 2m + 4}, \tilde{\nu} = \max\{1/m, \ (2\beta - m)/m^2\}$$

and ε is an arbitraryly smallpositive number.

Remark. When $m = 0$, $(1+t)^{-1/m}$ should be replaced by e^{-kt} with some $k > 0$.

Case 2: $\sigma(|\nabla u|^2) = |\nabla u|^m, m \geq 0$, and $g(u, \nabla u) = \pm|\nabla u|^{\beta+1}, \beta > m$.

In this case we can prove :

Theorem 3 *4 Let $p_0 \geq max\{m+2, N(\beta - m)\}$ Then, under Hyp.A,there exists $\varepsilon_0 > 0$ such that if $u_0 \in W_0^{1,p_0}(\Omega)$ and $\|\nabla u_0\|_{p_0} < \varepsilon_0$, then the problem (1.1)-(1.2) admits a solution $u(t)$ in the class*

$$L^\infty_{loc}\left((0,\infty); W_0^{1,\infty}(\Omega)\right) \cap W^{1,2}_{loc}\left((0,\infty); L^2(\Omega)\right) \cap L^\infty\left(R^+; W_0^{1,p_0}(\Omega)\right),$$

satisfying

$$\|\nabla u(t)\|_\infty \leq C(\|\nabla u_0\|_{p_0})\left(1 + t^{-N/(2p_0+mN)}\right)(1+t)^{-1/m}, \quad t > 0.$$

Remark. (1) In Theorem 1, uniqueness is open. (2) We have further

$$\left|\frac{\partial u}{\partial x_i}\right|^{m/2}\frac{\partial u}{\partial x_i}\in W^{1,2}_{loc}\left((0,\infty);L^2(\Omega)\right)\cap L^2_{loc}\left((0,\infty;H_1(\Omega)\right),\quad i=1,2,\cdots,N.$$

Case 3: $\sigma(|\nabla|^2)=\frac{1}{\sqrt{1+|\nabla u|^2}}$ and $g(u,\nabla u)=\pm|\nabla u|^{1+\beta},\beta>0$.

In this case our principal term is often called as 'mean curvature type'. As is mentioned in the introduction this is not coercive in the sense that

$$\lim_{\|\nabla u\|_2\to\infty}\frac{\int_\Omega\sigma(|\nabla u|^2)|\nabla u|^2dx}{\|\nabla u\|_2^2}\neq\infty$$

and the treatment of our strong perturbation is very delicate. We have the following result.

Theorem 4 *[5] Let $p_0>\max\{N(3+\alpha),2(m+1),\ 2\alpha+5\}$. Then, under Hyp.A there exists $\varepsilon_0>0$ such that if $u_0\in W_0^{1,p_0}(\Omega)$ and $\|\nabla u_0\|_{p_0}<\varepsilon_0$, then the problem (1.1)-(1.2) admits a unique solution $u(t)$ in the class*

$$L^\infty\left(R^+;W_0^{1,p_0}(\Omega)\right)\cap W^{1,2}\left(R^+;L^2(\Omega)\right)\cap L^2\left(R^+;W^{2,1+\kappa}(\Omega)\right)\cap L^\infty\left((0,\infty);W_0^{1,\infty}(\Omega)\right),$$

$(\kappa=(p_0-3)/(p_0+3))$, satisfying the estimates

$$\|\nabla u(t)\|_{p_0}\le C\|\nabla u_0\|_{p_0}e^{-\lambda_0 t},\quad t\ge 0,$$

$$\int_t^\infty\|u_t(s)\|^2\,ds\le C(\|\nabla u_0\|_{p_0})\,e^{-2\lambda_0 t}$$

$$\int_0^\infty\|u(t)\|^2_{2,1+\kappa}\,dt\le C(\|\nabla u_0\|_{p_0})<\infty$$

and

$$\|\nabla u(t)\|_\infty\le C(\|\nabla u_0\|_{p_0})\,t^{-N/(2p_0-3N)}\,e^{-\lambda_0 t},0<t<\infty,$$

where C denotes general constants independent of $u(t)$, $C(\|\nabla u_0\|_{p_0})$ denotes constants depending on $\|\nabla u_0\|_{p_0}$ and $\lambda_0=\lambda_0(\varepsilon_0-\|\nabla u_0\|_{p_0})>0$.

3 Outline of the proofs of Theorems

For the proofs of Theorems we derive apriori estimates for assumed smooth solutions $u(t)$, which will be sufficient for our purpose by limiting procedure of suitable approximate solutions.

Outline of the proof of Theorem 1

Multiplying the equation by $|u|^{q-2}u$ ($sign_o(u)$ if $q=1$) we have easily

$$\|u(t)\|_q \le \|u_0\|_q, 0 < t < \infty, \tag{3.1}$$

where we note

$$\int_\Omega \mathbf{b}(u)\cdot \nabla u |u|^{q-2} u dx = 0, q \ge 1$$

which comes from a special nonlinearity of convection.

Similarly, multiplying the equation by $|u|^{p-2}u$ we have

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p + \frac{C}{p}\| |u|^{(p+m)/(m+2)}\|_{1,m+2}^{m+2} \le 0. \tag{3.2}$$

Here, by Gagliardo-Nirenberg inequality, we see

$$\|u\|_p \le C^{(m+2)/(p+m)} \|u\|_q^{1-\theta} \| |u|^{(p+m)/(m+2)} \|^{(m+2)\theta/(p+m)}$$

with

$$\theta = \frac{m+2}{p+m}\cdot\frac{q^{-1}-p^{-1}}{N^{-1}-(m+2)^{-1}+(p+m)^{-1}(m+2)q^{-1}}.$$

Combining this with (3.1) and (3.2) we have

$$\frac{d}{dt}\|u(t)\|_p^p + C_p \|u(t)\|_p^{p(1+\mu_p^{-1})} \le 0$$

and hence,

$$\|u(t)\|_p \le C_p t^{-\lambda_p}, 0 < t \le 1$$

with

$$\lambda_p = \frac{N(p-q)}{p(q(m+2)+mN)}, \mu_p = p\lambda_p.$$

As is conjectured from this estimate , we apply Moser's technique to prove the estimate

$$\|u(t)\|_\infty \le Ct^{-\lambda}, 0 < t \le 1$$

where we recall $\lambda = N/(q(m+2)+mN)$.

More easily we can prove

$$\|u(t)\|_\infty \le C(1+t)^{-1/m}, \quad t \ge 1.$$

Next, multiplying the equation by u_t and u, respectively, and combining the resulting identities we can prove

$$\frac{d}{dt}\Gamma(t) + C\|u_t(t)\|_2^{-2}\Gamma^2(t) \le \int_\Omega |u|^{2\beta}|\nabla u|^2\, dx, \tag{3.3}$$

where

$$\Gamma(t) = \int_\Omega \int_0^{|\nabla u(t)\|^2} \sigma(s) ds dx.$$

Since

$$\|u(t)\|_2 \le Ct^{-\lambda(2-q)^+}$$

and

$$\int_\Omega |u|^{2\beta}|\nabla u|^2 dx \le Ct^{-\alpha}\Gamma(t)$$

we obtain from (3.3)

$$\frac{d}{dt}\Gamma(t) + Ct^{\lambda(2-q)^+}\Gamma^2(t) \le Ct^{-\alpha}\Gamma(t), \tag{3.4}$$

which gives the desired estimate for $\|\nabla u(t)\|_{m+2}, 0 < t \le 1$.

More easily, we can prove the desired estimate for $\|\nabla u(t)\|_{m+2}, t \ge 1$.

We see also

$$\int_t^T \|u_t(s)\|_2^2\, ds \le C_0(T)t^{-\tilde{\gamma}} \quad 0 < t \le T$$

with $\gamma = \mu(m+2) + (\alpha - 1)^+$.

A standard argument gives further

$$\|u_1(t) - u_2(t)\|_1 \le \|u_1(0) - u_2(0)\|_1$$

for two assumed solutions $u_1(t), u_2(t)$, which proves the uniqueness. Similarly, applying this to suitable approximate solutions $u_\varepsilon(t)$ we can see that $u_\varepsilon(t)$ converges to $u(t)$ unifromly in $L^1(\Omega)$.

Applying monotonicity argument we can prove that the limit function $u(t)$ is a desired solution.

Outline of the proof of Theorem 2

To prove the estimate for $\|\nabla u(t)\|_\infty$ we multiply the equation by by $-div\{|\nabla u|^{p-2}\nabla u\}, p \ge m+2$, and integrate by parts to get

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{k_0}{2}\int_\Omega |\nabla u|^{p+m-2}|D^2 u|^2\, dx + \frac{k_0(p-2)}{4}\int_\Omega |\nabla u|^{p+m-4}\nabla(|\nabla u|^2)|^2 dx$$

$$-(N-1)\int_{\partial\Omega} H(x)|\nabla u|^{p+m}\, dS \le \int_\Omega Cp^2 \int_\Omega |u|^{2\beta}|\nabla u|^{p-m}\, dx \tag{3.5}$$

and , by our assumption $H(x) \le 0$ (see [6]),

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{C_1}{p}\||\nabla u|^{(p+m)/2}\|_{1,2}^2 \le C_2 p^2 \int_\Omega |u|^{2\beta}|\nabla u|^{p-m}\, dx \tag{3.6}$$

with some $C_1, C_2 > 0$ independent of $p, p \ge m+4$. (When N=1 a modification is needed.)

Let $p_1 = m+2$ and we define a sequence $\{p_n\}$ by

$$p_n = 2p_{n-1} - m.$$

Then, by Gagliardo-Nirenberg inequality, we have

$$\|\nabla u\|_{p_n} \leq C^{2/(p_n+m)}\{\|\nabla u\|_{p_{n-1}}^{1-\theta_n}\||\nabla u|^{(p_n+m)/2}\|_{1,2}^{2\theta_n/(p_n+m)} + 1\}$$

with $\theta_n = N(1 - m/p_n)/(N+2)$. From this we can prove that

$$\|\nabla u(t)\|_{p_n} \leq \eta_n t^{-\xi_n} \quad ,0 < t \leq 1. \tag{3.8}$$

with $\xi_1 = \mu$ and ξ_n defined by

$$\xi_n = \frac{(p_n+m)(1-\theta)\xi_{n-1}}{p_n + m - p_n\theta_n}$$

$$+\max\{\frac{\theta_n}{p_n + m - \theta_n p_n}, \frac{\alpha(p_n+m)}{p_n(p_n+m-\theta_n p_n)} - \frac{1}{p_n}\}$$

$$= \frac{1}{p_n + m - p_n\theta_n}\{((p_n+m)(1-\theta_n)\xi_{n-1} + \theta_n) + \max\{0,\ (p_n+m)(\alpha-1)/p_n\}\}. \tag{3.9}$$

η_n is defined by

$$\eta_n = \{(2A_n)^{-p_n/\beta_n}(1 + (p_n+m)(\theta_n^{-1} - 1)\xi_{n-1})^{p_n/\beta_n}$$

$$+2C_n\{1 + (p_n+m)(\theta_n^{-1} - 1)\xi_{n-1}\}^{-1}\eta_{n-1}^{p_n(p_n+m)(1-\theta_n)/(p_n+m-p_n\theta_n)}\}^{1/p_n}$$

with certain constants A_n, B_n and C_n depending on p_n.

We can prove that $\{\eta_n\}$ is bounded and

$$\lim_{n\to\infty}\xi_n = \xi$$

(under some conditions on β), which proves the estimate for $0 < t \leq 1$ in Theorem 2. Similarly, we can prove the estimate of $\|\nabla u(t)\|_\infty$ for $t \geq 1$.

Outline of the proof of Theorem 3

For the equation considered in Theorem 3 we have the inequality (3.5) with the right hand side is replaced by $Cp^2 \int_\Omega |\nabla u|^{2\beta+p-m}\,dx$.

Further, if we assume $p_0 \geq \max\{N(\beta - m), 2\}$ we can prove

$$\frac{1}{p_0}\frac{d}{dt}\|\nabla u(t)\|_{p_0}^{p_0} + C_0\|w_0(t)\|_{H_1}^2 \leq C_1 p_0^2 \|\nabla u(t)\|_{p_0}^{2(\beta-m)}\|w_0(t)\|_{H_1}^2 \tag{3.9}$$

where $w_0(t) = |\nabla u|^{(m+p_0)/2}$. This inequality implies that

$$\|\nabla u(t)\|_{p_0}^{p_0} \leq \|\nabla u(t)\|_{p_0}, t \geq 0, \tag{3.10}$$

under the assumption

$$\|\nabla u_0\|_{p_0} \leq \varepsilon_0 \equiv (\frac{C_0}{C_1 p_0^2})^{1/2(p_0-m)}.$$

On the basis of the inequalities (3.9) and (3.10) we use Moser's technique to prove
$$\|\nabla u(t)\|_\infty \le Ct^{-N/(2p_0+mN)}, 0 < t \le 1.$$

Similarly, we obtain the desired estimate for $\|\nabla u(t)\|_\infty, t \ge 1$.

To show the convergence of appropriate approximate solutions $u^\varepsilon(t)$, $\varepsilon > 0$, to a desired solution $u(t)$ we must establish further a priori estimates including some second order derivatives of $u(t)$, which will assure the convergence

$$g(\nabla u_\varepsilon(t)) \to g(\nabla u(t)) \text{ in } L^p_{loc}((0,\infty); L^p(\Omega)), p \ge 2.$$

The following estimates are rather easily derived:

$$\int_t^T \int_\Omega |\nabla u(s)|^{2m} |D^2 u(s)|^2 dx ds \le C(\|\nabla u_0\|_{p_0}) t^{-(2\beta+2-m)\mu+1}, \mu = N/(2p_0 + mN), \tag{3.11}$$

and
$$\int_t^T \int_\Omega |u_t(s)|^2 dx ds \le C(\|\nabla u_0\|_{p_0}, T) t^{-(2\beta+2-p_o)\mu+1}.$$

Multiplying the equation by $-\frac{\partial}{\partial t}\{div(|\nabla u(t)|^m \nabla u(t)) - g(\nabla u)\}$ and using the above estimates we can prove that

$$\int_\varepsilon^T \int_\Omega |\nabla u(s)|^m |\nabla u_t(s)|^2 dx ds \le C(\|\nabla u_0\|_{p_0}, \varepsilon, T) < \infty, \tag{3.12}$$

for any $\varepsilon < T$.

(3.11) and (3.12) are sufficient for our purpose.

Outline of the proof of Theorem 4

In this case of the mean curvature type nonlinearity we obtain , instead of (3.6),

$$\frac{1}{p}\frac{d}{dt}\|\nabla u(t)\|_p^p + \frac{C_0}{p}\|F(|\nabla u|^2)\|_{H_1}^2 \le Cp^2 \left(\|\nabla u\|_p^{2\beta} + \|\nabla u\|_p^{2(\beta+3)}\right) \|F(|\nabla u|^2)\|_{H_1}^2 \tag{3.13}$$

provided that $p \ge N(\beta + 3)$, where we set

$$F(v) = p\int_0^v (1+\eta)^{-3/4} \eta^{(p-4)/4} d\eta.$$

We fix $p_0 \ge N(\alpha + 3)$ and write $F_0(t)$ for $F(t)$with $p = p_0$. Then, we have

$$\frac{d}{dt}\|\nabla u(t)\|_{p_0}^{p_0} \le \left\{-C_0 + p_0^3 C \left(\|\nabla u\|_{p_0}^{2\beta} + \|\nabla u\|_{p_0}^{2(\beta+3)}\right)\right\} \|F_0(|\nabla u|^2)\|_{H_1}^2$$

From this we conclude

$$\|\nabla u(t)\|_{p_0} \le \|\nabla u_{p_0}\| e^{-\lambda_0 t}, \lambda_0 > 0,$$

if $\|\nabla u_0\|_{p_0} < \varepsilon_0$ for some ε.

Noting

$$\int_\Omega |D^2u|^{1+\kappa}\,dx \le \int_\Omega \left\{(1+|\nabla u|^2)^{-3/2}\,|D^2u|^2\right\}^{(1+\kappa)/2}(1+|\nabla u|^2)^{3(1+\kappa)/4}\,dx$$

$$\le \left\{\int_\Omega (1+|\nabla u|^2)^{-3/2}|D^2u|^2\,dx\right\}^{(1+\kappa)/2}\left\{\int_\Omega (1+|\nabla u|^2)^{3(1+\kappa)/2(1-\kappa)}\,dx\right\}^{(1-\kappa)/2}$$

$$\le C(\|\nabla u_0\|_{p_0})\left\{\int_\Omega (1+|\nabla u|^2)^{-3/2}\,|D^2u|^2\,dx\right\}^{(1+\kappa)/2}$$

with $\kappa = (p_0-3)/(p_0+3)$, we obtain

$$\frac{d}{dt}\|\nabla u(t)\|_p^p + C_0\||\nabla u|^{p/2}\|_{1,1+\kappa}^2 \le C(\|\nabla u_0\|_{p_0})\,p^{\lambda+1}\|\nabla u(t)\|_p^p \tag{3.14}$$

for some $\lambda > 0$ independent of p. Applying Moser's technique to (3.14) we can derive the desired estimates in Theorem 4 for $\|\nabla u(t)\|_\infty$. Once the local boundedness of $\|\nabla u(t)\|_\infty$ is established the convergence of suitable approximate solutions to the solution is easier than $m-$ *Laplacian* case.

An open problem

In Theorems 3,4 we assumed that the initial data u_0 belong to W_0^{1,p_0} for some $p_0 > 0$ and $\|\nabla u\|_{p_0}$ are small , while in Therems 1, 2 we require only $u_0 \in L^q, q \ge 1$. It seems interesting problem to show global existence and some smoothing effect to the equation

$$u_t - \Delta u = |\nabla u|^\beta, \beta > 1,$$

with initial data $u_0 \in L^q$ with some $q, q \ge 1$.

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